# Hopf algebra structures of Multiple Zeta Values in positive characteristics

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# Outline

# Multiple Zeta Values(MZV's): classical setting

## Zeta values

Let $n \geq 2$ be an integer. The zeta value $\zeta(n) \in \mathbb{R}$ is defined by

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}.$$

# Multiple Zeta Values(MZV's): classical setting

## Zeta values

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$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}.$$

Even zeta values are known in terms of Bernoulli numbers, $\zeta(2) = \frac{\pi^2}{6}, \ \ldots, \ \zeta(2n) = \frac{B_{2n}}{2(2n)!}(2\pi)^{2n}$. In particular, $\frac{\zeta(2n)}{\pi^{2n}} \in \mathbb{Q}$.

Not much is known about odd zeta values; what we know is not much more than the following theorems:

Theorem (Apéry, 1978): $\zeta(3) \notin \mathbb{Q}$.
Theorem (Zudilin, 2001): $\{\zeta(5), \zeta(7), \zeta(9), \zeta(11)\} \not\subset \mathbb{Q}$.

# Multiple Zeta Values(MZV's): classical setting

Multiple Zeta Values (MZV's) generalize the zeta values.

## Multiple Zeta Values

Let $n_1, \ldots, n_{r-1} \geq 1$, $n_r \geq 2$ be integers.

$$\zeta(n_1, \ldots, n_r) := \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \in \mathbb{R}.$$

The *weight* and *depth* of the presentation $\zeta(n_1, \ldots, n_r)$ are $n_1 + \cdots + n_r$ and $r$, respectively. For example, $\zeta(4, 3)$ has weight 7 and depth 2.

MZV's are first introduced by Euler (as double zeta values) in 18c and Zagier in early 1990's.

# Multiple Zeta Values(MZV's): classical setting

Let $\mathcal{Z}$ the $\mathbb{Q}$-vector space spanned by all MZV's, and $\mathcal{Z}_w$ be the $\mathbb{Q}$-vector space spanned by the MZV's of weight $w$. (We let $\mathcal{Z}_0 = \mathbb{Q}$, $\mathcal{Z}_1 = \{0\}$.)

It is known that the product of two MZV's can be written as a $\mathbb{Q}$-linear combinations of MZV's. For example,

**Theorem (Euler, 1776.)**

For $n, m > 1$, $\zeta(n)\zeta(m) = \zeta(n+m) + \zeta(n, m) + \zeta(m, n)$.

This turns the vector space $\mathcal{Z}$ into an algebra, and the linear relations of MZV's are to be studied.

# Multiple Zeta Values(MZV's): classical setting

## Zagier-Hoffman Conjectures

(Zagier's conjecture) Let $(d_n)_{n\geq 0}$ be a sequence with $(d_0, d_1, d_2) = (1, 0, 1)$ and $d_n = d_{n-3} + d_{n-2}$ for $n \geq 3$. Then

$$\dim_{\mathbb{Q}} \mathcal{Z}_w = d_w \quad \text{for all } w \geq 0.$$

(Hoffman's conjecture) Further, and $\mathcal{Z}_w$ is spanned by

$$\{\zeta(k_1, \ldots, k_r) : k_1 + \cdots + k_r = w,\ 2 \leq k_i \leq 3\}.$$

Example. Zagier's conjecture implies $\zeta(1,2)/\zeta(3) \in \mathbb{Q}$. Indeed, $\zeta(1,2) = \zeta(3)$.

# MZV's in positive characteristics

Now we define MZV's in positive characteristics. Let

- $\mathbb{F}_q$ be a finite field of $q$ elements with characteristic $p > 0$,
- $A = \mathbb{F}_q[\theta]$, $A_+$ the set of monic polynomials in $A$,
- $K = \mathbb{F}_q(\theta)$, $K_\infty$ be the completion of $K$ at $\infty$.

**MZV's in positive characteristics (Carlitz 1935, Thakur 2004)**

Let $s_1, \ldots, s_r \geq 1$ be integers. The MZV in positive characteristics is defined by

$$\zeta_A(s_1, \ldots, s_r) := \sum \frac{1}{a_1^{s_1} \ldots a_r^{s_r}} \in K_\infty$$

where the sum is over $a_1, \ldots, a_r \in A_+$ and $\deg(a_1) > \cdots > \deg(a_r)$. In this presentation, weight and depth are $s_1 + \cdots + s_r$ and $r$, resp.

From now on, we denote $\mathcal{Z}_w$ the $K$-vector space spanned by MZV's of weight $w$. (so please forget the classical MZV's)

# Zagier-Hoffman conjectures in positive characteristics

Zagier-Hoffman conjectures were proved in positive characteristic case.

**Zagier-Hoffman conj. in pos. char. (Im-K.-Le-Ngo Dac-Pham)**

Let $(d_n)_{n\geq 0}$ be a sequence with $d_0 = 1$, $d_w = 2^{w-1}$ for $1 \leq w < q$, $d_q = 2^{q-1} - 1$, and $d(w) = \sum_{i=1}^{q} d(w-i)$ for $w > q$. Then,

$$\dim_K \mathcal{Z}_w = d_w \quad \text{for all } w \geq 0.$$

Further, we can exhibit a Hoffman-like basis of $\mathcal{Z}_w$.

This was proved for $w < 2q - 1$ by Ngo Dac, and for all $w$ by Im-K.-Le-Ngo Dac-Pham. (Also, by Chang-Chen-Mishiba independently).

# Outline

# Composition space and shuffle product

Let

- $\Sigma = \{x_n\}_{n\in\mathbb{N}}$ be a set of 'letters',
- $\langle\Sigma\rangle = \{x_{n_1}\dots x_{n_r} : x_{n_i} \in \Sigma \text{ for } r \geq 0\}$ be set of 'words' over $\Sigma$ with the empty word denoted by 1,
- $\mathfrak{C} = \mathbb{F}_q\langle\Sigma\rangle$ be the $\mathbb{F}_q$-vector space with basis $\langle\Sigma\rangle$, endowed with the *concatenation product* $\cdot$ (which can be omitted)

$$(x_{n_1}\dots x_{n_r})\cdot(x_{m_1}\dots x_{m_s}) = x_{n_1}\dots x_{n_r}x_{m_1}\dots x_{m_s}.$$

The weight and depth of $x_{n_1}\dots x_{n_r}$ are $n_1+\cdots+n_r$ and $r$, resp. For each nonempty $\mathfrak{a}\in\langle\Sigma\rangle$, we can write $\mathfrak{a} = x_a\cdot\mathfrak{a}_-$.

Later, we identify $\zeta_A(n_1,\dots,n_r)$ and $x_{n_1}\dots x_{n_r}$.

# Composition space and shuffle product

There is the notion of *shuffle product* in $\mathfrak{C}$ defined by Chen's identity: $\mathfrak{u} \sqcup\!\sqcup 1 = 1 \sqcup\!\sqcup \mathfrak{u} = \mathfrak{u}$ for $\mathfrak{u} \in \langle \Sigma \rangle$, and for nontrivial $\mathfrak{a}$ and $\mathfrak{b}$

$$\mathfrak{a} \sqcup\!\sqcup \mathfrak{b} := x_a(\mathfrak{a}_- \sqcup\!\sqcup \mathfrak{b}) + x_b(\mathfrak{a} \sqcup\!\sqcup \mathfrak{b}_-) + x_{a+b}(\mathfrak{a}_- \sqcup\!\sqcup \mathfrak{b}_-) + \sum_{0<j<a+b} \Delta^j_{a,b} x_{a+b-j} \cdot (x_j \sqcup\!\sqcup (\mathfrak{a}_- \sqcup\!\sqcup \mathfrak{b}_-)).$$

Here $\Delta^j_{a,b} = (-1)^{a-1}\binom{j-1}{a-1} + (-1)^{b-1}\binom{j-1}{b-1} \in \mathbb{F}_q$ when $(q-1) \mid j$, and $\Delta^j_{a,b} = 0$ otherwise.

N.B. the Chen's identity (of depth one version),

$$\zeta_A(a)\zeta_A(b) = \zeta_A(a,b) + \zeta_A(b,a) + \zeta_A(a+b) + \sum_{0<j<a+b} \Delta^j_{a,b}\zeta_A(a+b-j,j) ;$$

$\sqcup\!\sqcup$ in $\mathfrak{C}$ is defined to satisfy $\zeta(\mathfrak{a} \sqcup\!\sqcup \mathfrak{b}) = \zeta(\mathfrak{a}) \times \zeta(\mathfrak{b}) \in K_\infty$ when we identify $\zeta(x_{n_1} \dots x_{n_r})$ and $\zeta_A(n_1, \dots, n_r)$.

Note that $\sqcup\!\sqcup$ preserves the weight; i.e. $\mathrm{w}(\mathfrak{a} \sqcup\!\sqcup \mathfrak{b}) = \mathrm{w}(\mathfrak{a}) + \mathrm{w}(\mathfrak{b})$.

# Hopf algebra

In her thesis, Shuhui Shi (2015) proposed that the MZV's in positive characteristics have a Hopf algebra structure with the shuffle product $\sqcup\!\sqcup$ and the coproduct $\Delta_{Shi}$ (which will be defined later).

Before proceeding ahead, we introduce a brief notion of Hopf algebra.

# (But what is Hopf algebra?)

Hopf algebra is an algebraic structure arising in many areas of mathematics, including algebraic topology, representation theory, and combinatorics.

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Hopf algebra is an algebraic structure arising in many areas of mathematics, including algebraic topology, representation theory, and combinatorics.

A bialgebra over a field $k$ is a $k$-vector space which is both (co)unital (co)associative algebra and coalgebra, with compatibilities between two structures.
In other words, it is a quintuple $(A, M, u, \Delta, \epsilon)$, where

- $A$ is a $k$-vector space,
- $M\colon A \otimes A \to A$ the product; we write $M(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} * \mathfrak{b}$,
- $u\colon k \to A$ the unit map,
- $\Delta\colon A \to A \otimes A$ the coproduct, and
- $\epsilon\colon A \to k$ the counit map (or augmentation map),

with the following properties (next slide).

# (But what is Hopf algebra?)

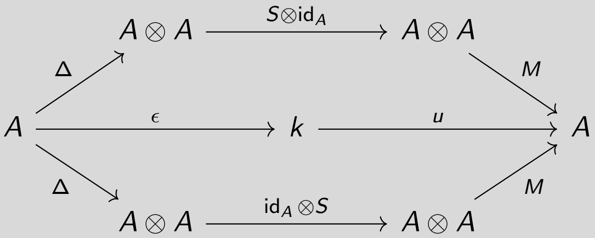
Bialgebra axioms are as follows:

- associativity, $M \circ (M \otimes \mathrm{id}) = M \circ (\mathrm{id} \otimes M)$, i.e. $(\mathfrak{a} * \mathfrak{b}) * \mathfrak{c} = \mathfrak{a} * (\mathfrak{b} * \mathfrak{c})$ for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A$,
- unitary property, i.e. there exist $I \in A$ with $I * \mathfrak{a} = \mathfrak{a} * I = \mathfrak{a}$. The unit map will be given as $u(f) = f \cdot I$,
- coassociativity, $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$
- counitary, $(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$; This can be understood as the counit map $\epsilon$ collapses (or 'undo') the extra structure from the coproduct $\Delta$ on the both sides and recover the original element.
- compatibilities for $M$ and $\Delta$, $u$ and $\Delta$, $M$ and $\epsilon$, and $u$ and $\epsilon$. i.e. $\Delta(\mathfrak{a} * \mathfrak{b}) = \Delta(\mathfrak{a}) * \Delta(\mathfrak{b})$, $\Delta(I) = I \otimes I$ (where $I = u(1)$), $\epsilon(\mathfrak{a} * \mathfrak{b}) = \epsilon(\mathfrak{a})\epsilon(\mathfrak{b})$, and $\epsilon(I) = 1$.

(In $A \otimes A$, $(a_1 \otimes a_2) * (b_1 \otimes b_2) := (a_1 * b_1) \otimes (a_2 * b_2)$.)

# (But what is Hopf algebra?)

A bialgebra $(A, M, u, \Delta, \epsilon)$ is said to be a Hopf algebra if the antipode map $S\colon A \to A$ exists, satisfying the following commutative diagram:




($S$ can be understood as an 'inverse element' of $\mathrm{id}_A\colon A \to A$ in $\mathrm{Hom}(A, A)$ wrt. the convolution product $f \star g := M \circ (f \otimes g) \circ \Delta$.)

# Example of Hopf algebra: Group algebra

Let $k$ a field and $G$ be a (finite) group, and $kG$ be the group algebra. Then $kG$ is a Hopf algebra with the following structure:

- $\Delta(g) = g \otimes g$,
- $u(a) = a1_G$,
- $\epsilon(g) = 1_k$, and
- $S(g) = g^{-1}$ for all $g \in G$.

# Example of Hopf algebra: Shuffle algebra

Let $k$ a field and $\Sigma = \{x_n\}_{n\in\mathbb{N}}$.
Let $\langle\Sigma\rangle = \{x_{n_1}\dots x_{n_r} : x_{n_i} \in X \text{ for } r \geq 0\}$ be the set of words over $X$ with the empty word 1 and the concatenation '$\cdot$'.
Let $\mathfrak{S} = k\langle X\rangle$ be the $k$-vector space with basis $\langle X\rangle$, endowed with the shuffle product $*$ defined as

$$1 * w = w * 1 = w \quad (\forall w \in \langle X\rangle),$$
$$(x_a\mathfrak{a}_-) * (x_b\mathfrak{b}_-) = x_a \cdot (\mathfrak{a}_- * \mathfrak{b}) + x_b \cdot (\mathfrak{a} * \mathfrak{b}_-).$$

Then $\mathfrak{S}$ is a Hopf algebra with the 'de-concatenation' coproduct

$$\Delta_{deconcat}(w) := \sum_{uv=w} u \otimes v,$$

e.g. $\Delta_{deconcat}(xyz) = 1 \otimes xyz + x \otimes yz + xy \otimes z + xyz \otimes 1.$

# Hopf algebra structure of MZV's in positive characteristics

Shi (2015) suggested the definition of coproduct $\Delta_{Shi}$ compatible to the shuffle product $\sqcup\!\sqcup$ in $\mathfrak{C}$.
She then proved that $\mathfrak{C}$ has a Hopf algebra structure, under the assumptions of (1) the associativity of $\sqcup\!\sqcup$, (2) the coassociativity of $\Delta_{Shi}$, and (3) the compatibility of $\sqcup\!\sqcup$ and $\Delta_{Shi}$.

We (Im-Kim-Le-Ngo Dac-Pham, 2023) proved that $\mathfrak{C}$ is indeed a Hopf algebra with $\sqcup\!\sqcup$ and $\Delta_{Shi}$.

# Outline

# Shi's construction of the coproduct

Shi gave the inductive definition of the coproduct $\Delta_{Shi}$ on $\mathfrak{C}$.

$$\Delta_{Shi}(1) := 1 \otimes 1, \quad \Delta_{Shi}(x_1) := 1 \otimes x_1 + x_1 \otimes 1. \quad \text{(initial cases)}$$

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Now assume that we've defined all $\Delta(\mathfrak{u})$ of weight$(\mathfrak{u}) < w$. First, for a word $\mathfrak{a} = x_a \mathfrak{a}_-$ with weight $w$ and depth $> 1$ with

$$\Delta_{Shi}(x_a) =: 1 \otimes x_a + \sum \mathfrak{a}_1 \otimes \mathfrak{a}_2,$$

$$\Delta_{Shi}(\mathfrak{a}_-) =: \sum \mathfrak{u}_1 \otimes \mathfrak{u}_2, \quad \text{(known by the induction hypothesis)}$$

Shi defined

$$\Delta_{Shi}(x_a \mathfrak{a}_-) := 1 \otimes \mathfrak{a} + \sum (\mathfrak{a}_1 \cdot \mathfrak{u}_1) \otimes (\mathfrak{a}_2 \sqcup \mathfrak{u}_2).$$

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Finally, Shi defined $\Delta_{Shi}(x_w)$ to satisfy

$$\Delta_{Shi}(x_1 ⧢ x_{w-1}) = \Delta_{Shi}(x_1) ⧢ \Delta_{Shi}(x_{w-1}).$$

Note that $x_1 ⧢ x_{w-1} = x_w + (\text{other terms})$; the coproduct of all other terms are known in this step.

# Our construction of the coproduct

We introduce a different definition of coproduct $\Delta$ on $\mathfrak{C}$.
We first define $\triangleright$ on $\mathfrak{C}$ recursively. As initial cases we let

$$1 \triangleright \mathfrak{u} := \mathfrak{u} =: \mathfrak{u} \triangleright 1 \quad \text{for all } \mathfrak{u}.$$

For nontrivial word $\mathfrak{a} = x_a \mathfrak{a}_-$, we define

$$\mathfrak{a} \triangleright \mathfrak{b} := x_a \cdot (\mathfrak{a}_- \sqcup\!\sqcup \mathfrak{b}).$$

N.B. $x_a \triangleright \mathfrak{u} = x_a \cdot \mathfrak{u}$, but $\mathfrak{u} \triangleright \mathfrak{v} \neq \mathfrak{u} \cdot \mathfrak{v}$, and $\triangleright$ is not commutative nor associative in general.

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Construction of $\Delta$ is then similar to $\Delta_{Shi}$, but the concatenation in

$$\Delta_{Shi}(x_a \mathfrak{a}_-) := 1 \otimes \mathfrak{a} + \sum (\mathfrak{a}_1 \cdot \mathfrak{u}_1) \otimes (\mathfrak{a}_2 \sqcup\!\sqcup \mathfrak{u}_2)$$

is replaced by the triangle product, i.e.,

$$\Delta(x_a \mathfrak{a}_-) := 1 \otimes \mathfrak{a} + \sum (\mathfrak{a}_1 \triangleright \mathfrak{u}_1) \otimes (\mathfrak{a}_2 \sqcup\!\sqcup \mathfrak{u}_2).$$

# Our construction of the coproduct

Now we have two questions:

(Q1) Is it true that $\Delta = \Delta_{Shi}$?

(Q2) Does $\Delta$ satisfy the Hopf algebra axioms?

# Our construction of the coproduct

Now we have two questions:
(Q1) Is it true that $\Delta = \Delta_{Shi}$?
(Q2) Does $\Delta$ satisfy the Hopf algebra axioms?

We proved that $⧢$ is associative, and $\Delta$ satisfies the compatibility and coassociativity and properties, i.e.

$$(\mathfrak{a} ⧢ \mathfrak{b}) ⧢ \mathfrak{c} = \mathfrak{a} ⧢ (\mathfrak{b} ⧢ \mathfrak{c}),$$
$$\Delta(\mathfrak{u}) ⧢ \Delta(\mathfrak{v}) = \Delta(\mathfrak{u} ⧢ \mathfrak{v}), \quad (\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta.$$

Also, we proved that $\Delta(x_n) = \sum \mathfrak{u} \otimes \mathfrak{v}$ satisfies the condition $\mathrm{depth}(\mathfrak{u}) \leq 1$ for all $n \geq 1$.

With this we first answer (Q1), and then according to Shi's proof of the remaining Hopf algebra axioms for $\Delta_{Shi} = \Delta$, we answer (Q2).

# Algebra structure (Associativity of $\sqcup\!\sqcup$)

**Theorem.**

$\sqcup\!\sqcup$ is associative.
In particular, the space $(\mathfrak{C}, \sqcup\!\sqcup)$ is commutative $\mathbb{F}_q$-algebra with algebra homomorphism $Z_{\sqcup\!\sqcup} \colon K \otimes_{\mathbb{F}_q} \mathfrak{C} \to \mathcal{Z}$ given by $\mathfrak{a} \mapsto \zeta_A(\mathfrak{a})$.

The proof relies on huge amount of technical calculation. One of the key fact is

**Lemma (Partial Fractions)**

Let $r, s \in \mathbb{Z}_{\geq 1}$. As rational functions in $\mathbb{Q}(X, Y)$,

$$\frac{1}{X^r Y^s} = \sum_{\substack{i+j=r+s \\ i,j \in \mathbb{Z}_{\geq 0}}} \left( \binom{j-1}{s-1} \frac{1}{X^i (X+Y)^j} + \binom{j-1}{r-1} \frac{1}{Y^i (X+Y)^j} \right).$$

This is the fact which Chen used to find the coefficients $\Delta^j_{a,b}$.

# Algebra structure (Associativity of ⧢)

Let $r, s, t \geq 1$. We expanded two different partial fractions for

$$\frac{1}{A^r B^s} \cdot \frac{1}{C^t} = \frac{1}{A^r} \cdot \frac{1}{B^s C^t}.$$

For each $d \in \mathbb{Z}_{\geq 1}$, we partitioned the indices $(a, b, c) \in A_+^3(d)$ into
$M_0 = \{(a, b, c) : a = b = c\}$,
$M_1 = \{(a, b, c) : \text{only two are the same}\}$ and
$M_2 = \{(a, b, c) : a \neq b \neq c \neq a\}$ which is partitioned further into

- $N_0$ with $b - a = \lambda f$, $c - a = \mu f$, and $\lambda \neq \mu$,
- $N_1$ with $b - a = \lambda f$, $c - a = \mu u$,
- $N_2$ with $b - a = \mu u$, $c - a = \lambda f$,
- $N_3$ with $b - a = \lambda f$, $c - a = \lambda f + \mu u$,
- $N_4$ with $b - a = \lambda f$, $c - a = \mu f + \eta u$, and $\lambda \neq \mu$,

for some $\lambda, \mu, \eta \in \mathbb{F}_q^\times$ and $f, u \in A_+$ with $\deg(u) < \deg(f) < d$.

# Algebra structure (Associativity of $\sqcup\!\sqcup$)

By calculating and comparing the sums $\sum \frac{1}{a^r b^s} \cdot \frac{1}{c^t}$ and $\sum \frac{1}{a^r} \cdot \frac{1}{b^s c^t}$ over each partition, we deduce that the sums over

- $M_0$ induce the same expression of depth one MZV's,
- $M_1 \sqcup N_0$ induce the same expression of depth two MZV's, and
- $N_1 \sqcup N_2 \sqcup N_3 \sqcup N_4$ induce the same expression of depth three MZV's

of the associativity equation

$$(\zeta_A(r)\zeta_A(s))\zeta_A(t) = \zeta_A(r)(\zeta_A(s)\zeta_A(t))$$

in terms of power sums, which is translated into the associativity of $\sqcup\!\sqcup$ in $\mathfrak{C}$.

For general case we can proceed with the induction on the sum of depths.

# Algebra structure (Associativity of $ш$)

Example. Let $q = 3$. Chen's identity yields

$$\zeta_A(1) \cdot \zeta_A(1) = 2\zeta_A(1,1) + \zeta_A(2).$$

This is not only true as values in $K_\infty$, but also gives the equality of the elements in $\mathfrak{C}$, i.e.

$$x_1 \sqcup\!\sqcup\, x_1 = 2x_1x_1 + x_2.$$

By applying the Chen's identity again, we have

$$\begin{aligned}(\zeta_A(1) \cdot \zeta_A(1)) \cdot \zeta_A(2) &= 2\zeta_A(1,1,2) + 2\zeta_A(1,2,1) + 2\zeta_A(1,3) \\ &\quad + 2\zeta_A(2,1,1) + 2\zeta_A(3,1) + \zeta_A(4)\end{aligned}$$

$$\text{yields} \quad (x_1 \sqcup\!\sqcup\, x_1) \sqcup\!\sqcup\, x_2 = 2x_1x_1x_2 + 2x_1x_2x_1 + \cdots + 2x_3x_1 + x_4.$$

Further, as the expression calculated by Chen's identity for $\zeta_A(1) \cdot (\zeta_A(1) \cdot \zeta_A(2))$ is the same as the above, then we conclude that $(x_1 \sqcup\!\sqcup\, x_1) \sqcup\!\sqcup\, x_2 = x_1 \sqcup\!\sqcup\, (x_1 \sqcup\!\sqcup\, x_2)$.

# Hopf algebra structure (Axioms for coproduct $\Delta$)

Recall $1 \triangleright \mathfrak{a} = \mathfrak{a} \triangleright 1 = \mathfrak{a}$, and $\mathfrak{a} \triangleright \mathfrak{b} = x_a \cdot (\mathfrak{a}_- \sqcup\!\sqcup \mathfrak{b})$ for nonempty $\mathfrak{a}$. We define $\diamond$ on $\mathfrak{C}$ with $1 \diamond \mathfrak{a} = \mathfrak{a} \diamond 1 = \mathfrak{a}$, and for nonempty $\mathfrak{a}$ and $\mathfrak{b}$,

$$\mathfrak{a} \diamond \mathfrak{b} := x_{a+b}(\mathfrak{a}_- \sqcup\!\sqcup \mathfrak{b}_-) + \sum_{0<j<a+b} \Delta^j_{a,b} \cdot ((\mathfrak{a}_- \sqcup\!\sqcup \mathfrak{b}_-) \sqcup\!\sqcup x_j).$$

By introducing the new operators $\diamond$ and $\triangleright$ and the new definition for $\Delta$ (and another huge amount of calculations), we could prove the compatibility and coassociativity results. Some key lemmas follow.

## Lemmas

- $\mathfrak{a} \sqcup\!\sqcup \mathfrak{b} = \mathfrak{a} \diamond \mathfrak{b} + \mathfrak{a} \triangleright \mathfrak{b} + \mathfrak{b} \triangleright \mathfrak{a}$ (Definition),
- $\mathfrak{a} \diamond \mathfrak{b} = (x_a \diamond x_b) \triangleright (\mathfrak{a}_- \sqcup\!\sqcup \mathfrak{b}_-)$,
- $(\Delta(\mathfrak{u}) - 1 \otimes \mathfrak{u}) \triangleright \Delta(\mathfrak{v}) = \Delta(\mathfrak{u} \triangleright \mathfrak{v}) - 1 \otimes (\mathfrak{u} \triangleright \mathfrak{v})$, when $(\mathfrak{u}_1 \otimes \mathfrak{u}_2) \triangleright (\mathfrak{v}_1 \otimes \mathfrak{v}_2) := (\mathfrak{u}_1 \triangleright \mathfrak{v}_1) \otimes (\mathfrak{u}_2 \sqcup\!\sqcup \mathfrak{v}_2)$.

# Hopf algebra structure (Comparison to $\Delta_{Shi}$)

(Q1) is remaining: $\Delta = \Delta_{Shi}$?

We introduce braket operator, $[1] = 1$ and

$$[x_{n_1} \dots x_{n_r}] := \left( (-1)^r \cdot \Delta_{1,w+1}^{n_1} \cdots \Delta_{1,w+1}^{n_r} \right) (x_{n_1} ⧢ \dots ⧢ x_{n_r}).$$

N.B. $[\mathfrak{u}] = 0$ if $(q-1) \nmid \mathrm{weight}(\mathfrak{u})$, $[\mathfrak{a} \cdot \mathfrak{b}] := [\mathfrak{a}] ⧢ [\mathfrak{b}]$.

**Proposition**

$$\Delta(x_n) = 1 \otimes x_n + \sum_{\substack{r \in \mathbb{Z}_{\geq 1}, \mathfrak{a} \in \langle \Sigma \rangle \\ r + w(\mathfrak{a}) = n}} \binom{r + depth(\mathfrak{a}) - 2}{depth(\mathfrak{a})} x_r \otimes [\mathfrak{a}],$$

in particular, $\Delta(x_n) = 1 \otimes x_n + \sum \mathfrak{u} \otimes \mathfrak{v}$ with $\mathrm{depth}(\mathfrak{u}) = 1$ for all $n$.

**Proposition**

We have $\Delta = \Delta_{Shi}$.

# Hopf algebra structure

**Theorem (Im-K.-Le-Ngo Dac-Pham)**

$(\mathfrak{C}, ш, u, \Delta, \epsilon)$ is a connected graded Hopf algebra of finite type over $\mathbb{F}_q$.

# Outline

# Remark on the coproduct of letters

We also found some explicit formulae for $\Delta(x_n)$.

**Proposition**

When $n \leq q$, $\Delta(x_n) = 1 \otimes x_n + x_n \otimes 1$.
When $q < n \leq q^2$,

$$\Delta(x_n) = 1 \otimes x_n + x_n \otimes 1 + \sum_{i=1}^{k} (-1)^i \binom{n-1+i}{i} x_{n-i(q-1)} \otimes x_{i(q-1)}$$

when $k$ is integer with $kq < n \leq (k+1)q$.

You can find the numerical results for $\Delta(x_n)$ for $n \leq q^3 + q^2$ and $q = 3, 5$ cases in our paper.

# Remark on the stuffle Hopf algebra structure

Instead of $\sqcup\!\sqcup$ we can define the *stuffle product* $*$ as

$$1 * \mathfrak{a} = \mathfrak{a} * 1 = \mathfrak{a} \quad \text{for all } \mathfrak{a},$$
$$\mathfrak{a} * \mathfrak{b} = x_a(\mathfrak{a}_- * \mathfrak{b}) + x_b(\mathfrak{a} * \mathfrak{b}_-) + x_{a+b}(\mathfrak{a}_- * \mathfrak{b}_-) \quad \text{for nontrivial } \mathfrak{a}, \mathfrak{b}.$$

**Theorem**

$\mathfrak{C}$ with $*$ and coproduct $\Delta_{deconcat}$ attains the connected graded Hopf algebra of finite type over $\mathbb{F}_q$.

N.B. As stuffle algebra, $Z_* : \mathfrak{C} \otimes_{\mathbb{F}_q} K \to \mathcal{Z}$; $\mathfrak{a} \mapsto \mathrm{Li}(\mathfrak{a})$ is $K$-algebra homomorphism, where Li is the Carlitz multiple polylogarithms which spans the same space as the MZV's.

# Remark on the Alternating MZV's

Finally we remark that the Hopf algebra structure of the alternating MZV's (abbreviated as AMZV's) in positive characteristics is also proved in (Im-Kim-Le-Ngo Dac-Pham 2023a), where AMZV's are defined (Harada, 2021) as

$$\zeta_A\begin{pmatrix}\varepsilon_1 & \dots & \varepsilon_r \\ s_1 & \dots & s_r\end{pmatrix} := \sum \frac{\varepsilon_1^{\deg a_1} \dots \varepsilon_r^{\deg a_r}}{a_1^{s_1} \dots a_r^{s_r}}$$

for positive integers $s_i$'s and $\varepsilon_i \in \mathbb{F}_q^\times$, where the sum is over all monic polynomials $a_i$'s with $\deg(a_1) > \cdots > \deg(a_r)$, with the similarly defined shuffle product and coproduct.

## Thank you for your attention!

Thank you for your attention!
Questions are welcome!